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SUBSTITUTION-LIKE MINIMAL SETS

BY

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ABSTRACT

Substitution-like minimal sets are a class of symbolic minimal sets on two symbols which includes the discrete substitution minimal sets on two symbols. They are almost automorphic extensions of the n-adic integers and they are constructed by using special subsets of the n-adics from which the almost automorphic points are determined by following orbits in the n-adics. Through their study a complete classification is obtained for substitution minimal sets of constant length on two symbols. Moreover, the classification scheme is such that for specific substitutions the existence or non-existence of an isomorphism can be determined in a finite number of steps.

One natural way to construct symbolic minimal sets is to look for recursive processes which generate almost periodic sequences. This is how Morse constructed the sequence which now bears his name (see [11, pp. 95–96]). He was aware of the fact that his sequence was invariant under a substitution, and in Oldenburger's notes it is defined using the usual substitution [12, p. 24]. However, Gottschalk was the first to systematically study substitutions as a recursive method for constructing almost periodic sequences. The ideas which he developed in "Substitution minimal sets" [4] are already apparent in a slightly earlier paper of his [3]. Since then a number of people have contributed to the theory of substitution minimal sets including Keynes [6], Coven [1,2], Keane [2], Klein [7], and Martin [9,10].

The idea behind our work was to see if we could obtain some new results about almost automorphic substitution minimal sets by using the characteristic sequence point of view which allows the study of all of the almost automorphic sequences in a symbolic minimal set simultaneously [8, sect. 1]. If A is a subset of the group of n-adic integers, denoted by $Z(n)$, then we consider sequences of the form $x(k) = \chi_A(z + k\hat{1})$ where χ_A is the characteristic function of A, $\hat{1}$ is the canonical generator of $Z(n)$, k is an integer, and z is a fixed element of $Z(n)$. By imposing certain recursive properties on A we obtain sequences whose orbit closures have substitution-like properties. In other words, we are

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constructing almost periodic sequences which are indirectly defined by a recursive process. The central portion of this paper is devoted to the study of these substitution-like minimal sets.

In the last section we obtain two new results about substitution minimal sets on two symbols. First, most of them are prime extensions of their maximal equicontinuous factor; i.e., all their factors are equicontinuous. The exceptions to this have at most one non-equicontinuous factor. Second, two such substitutions θ and ϕ of length n and m which are in normal form in the sense of Coven and Keane produce isomorphic minimal sets if and only if there exist positive integers p and q such that $n^p = m^q$ and $\theta^p = \phi^q$. This is an extension of the classification theorem given by Coven and Keane [2], and in the course of proving our result we give a new proof of their theorem.

Although our work partially supersedes the work of Coven and Keane, we are indebted to them for many valuable clues. We also wish to thank E. Coven and M. Paul for helpful conversations while this work was in progress.

Introduction

Let $\Omega = \{0,1\}^Z$ with the product topology where Z denotes the integers. We denote the nth coordinate of $x \in \Omega$ by $x(n)$, and we define a homeomorphism σ of Ω onto Ω by $\sigma x(n) = x(n+1)$. The transformation group (Ω, σ) is called the shift dynamical system on two symbols. Within (Ω, σ) there is a vast array of minimal subsets.

The almost automorphic minimal subsets of (Ω, σ) can be constructed by the following process. Let G be a compact metric topological group written additively. We assume that G is monothetic; i.e., there exist $g \in G$, called generators, such that $\{ng : n \in Z\}$ is dense in G. Let (G, g) denote the minimal equicontinuous transformation group determined by $z \to z + g$. We define a collection of subsets of G as follows:

$$\mathscr{C}(G) = \{A \subset G : \mathrm{Cl}(\mathrm{Int}\, A) = A,\ \partial A \neq \emptyset, \text{and } A + z = A \Rightarrow z = 0\}.$$

(Cl, Int, ∂ denote the closure, interior and boundary operators respectively.) Fix $A \in \mathscr{C}(G)$ and g a generator of G and set $T_A = G \setminus \bigcup_{n=-\infty}^{\infty} \partial A + ng$ which is not empty by the Baire Category Theorem. We define an x in Ω by the formula

$$x(n) = \chi_A(z + ng)$$

where χ_A is the characteristic function of A and $z \in T_A$ and we call such an x a characteristic sequence. We will briefly recount the essential properties of these sequences. A complete discussion of them can be found in the first section of [8]. Let $X_A = \mathrm{Cl}\,\mathcal{O}(x))$ where $\mathcal{O}(x) = \{\sigma^n(x): n \in Z\}$ and denote σ restricted to X_A also by σ. Then (X_A, σ) is an almost automorphic minimal set which is independent of the choice of $z \in T_A$. Moreover, there exists a homomorphism p_A of (X_A, σ) onto (G, g) with the following properties for $x, y \in X_A$:

a) $p_A(x) = p_A(y)$ if and only if x and y are proximal.

b) x is a characteristic sequence if and only if $p_A(x) \in T_A$ if and only if $p_A^{-1}[p_A(x)] = \{x\}$.

c) If x is a characteristic sequence, then $x(n) = \chi_A(p_A(x) + ng)$. In particular $x(0) = 1$ if and only if $p_A(x) \in A$.

Every almost automorphic minimal subset of (Ω, σ) which is not a periodic orbit can be described in this way.

Throughout the rest of this paper we will assume that G is zero-dimensional. Characteristic sequences over zero-dimensional groups have been studied but under different names. A sequence $\omega \in \Omega$ is called a regularly almost periodic point of (Ω, σ) if given any neighborhood V of ω there exists a proper subgroup H of Z such that $\sigma^n(\omega) \in V$ for all n in H. Given $\omega \in \Omega$ suppose there exists a collection of pairwise disjoint arithmetic progressions $\{T_i\}$ whose union is Z such that $n, m \in T_i$ implies $\omega(n) = \omega(m)$, then ω is a Toeplitz sequence. Moreover, if the T_i can be chosen so that $\Sigma_i\, 1/q_i = 1$ where $T_i = \{r_i + kq_i : k \in Z\}$, then ω is a regular Toeplitz sequence. Toeplitz sequences were first investigated by Jacobs and Keane using different definitions [5]. The above definitions are from Coven and Keane [2] and it is easy to check that they are equivalent to the original ones.

PROPOSITION 1. *Let $x \in \Omega$. The following are equivalent:*

a) *x is a Toeplitz sequence.*

b) *x is a regularly almost periodic point.*

c) *x is a characteristic sequence constructed on a zero-dimensional group or a periodic sequence.*

PROOF. First let x be a Toeplitz sequence with arithmetic progressions $\{T_i\}$. We may as well assume that x is not periodic and the T_i are indexed by the positive integers. Let $T_i = \{r_i + kq_i : k \in Z\}$, let m_i be the least common multiple of $\{q_1, \cdots, q_i\}$ and let

$$B_i = \left\{\omega \in \Omega : \omega(j) = x(j)\,\forall\, j \in [-m_i, m_i] \cap \left(\bigcup_{k \le i} T_i\right)\right\}.$$

Then $\bigcap_{i=1}^{\infty} B_i = \{x\}$ because $m_i \to \infty$ as $i \to \infty$, $\sigma^k(x) \in B_i$ if m_i divides k, and x is regularly almost periodic.

Now let x be a regularly almost periodic point in (Ω, σ). The trace of x is $\{x\}$ and hence on $X = \mathrm{Cl}(\mathcal{O}(x))$ the equicontinuous structure relation coincides with trace relation and the proximal relation [4, theor. 1.08 (4)]. From this point it is easy to see that x is an almost automorphic point of (Ω, σ) and thus a characteristic sequence. The maximal equicontinuous factor of (X, σ) is zero-dimensional by [4, theor. 1.15 (5)].

Finally, let x be a characteristic sequence determined by $A \in \mathscr{C}(G)$ with G zero-dimensional, and let $\{U_n\}$ be a sequence of open-closed subgroups of G such that $\cap U_n = \{0\}$. Set $V_1 = \{z : U_1 + z \subset A\}$ and $W_1 = \{z : U_1 + z \subset G \setminus A\}$. If V_k and W_k have been defined, then set $V_{k+1} = \{z : U_{k+1} + z \subset A \setminus (\cup_{i=1}^{k} V_i)\}$ and $W_{k+1} = \{z : U_{k+1} + z \subset (G \setminus A) \setminus (\cup_{i=1}^{k} W_i)\}$. Observe that $\mathrm{Int}\, A = \cup_{i=1}^{\infty} V_i$ and $G \setminus A = \cup_{i=1}^{\infty} W_i$. Let $i(n)$ denote the index of U_n in G. Suppose that $z = p_A(x)$ and $z + mg \in U_n + w \subset A$ or $G \setminus A$. It then follows that $z + mg + ki(n)g \subset U_n + w$ for all k in Z because $U_n = \mathrm{Cl}(\{ki(n)g : k \in Z\})$. In other words, $x(m) = x(m + ki(n))$ for all $k \in Z$.

We will now define the arithmetic progressions for x inductively. First $z \in V_n$ or W_n for some unique n, call it n_1. Set $T_1 = \{ki(n_1) : k \in Z\}$. Choose m_2 with $|m_2|$ minimal and $m_2 \notin T_1$. There exists a unique n such that $z + m_2 g \in V_n$ or W_n, call it n_2, and set $T_2 = \{m_2 + ki(n_2) : k \in Z\}$. Continuing this way we define $\{T_i\}_{i=1}^{\infty}$ such that $x(m) = x(m')$ if $m, m' \in T_i$. It is easy to check that $\cup_{i=1}^{\infty} T_i = Z$ and $T_i \cap T_j = \varnothing$ if $i \neq j$.

The equivalence of a) and b) is not new. The interesting thing is the proof of c) implies a) because it constructs arithmetic progressions for a Toeplitz sequence x which are intimately connected with the structure of the maximal equicontinuous factor of $(\mathrm{Cl}(\mathcal{O}(x)), \sigma)$. Moreover, notice that our construction gives progressions such that $1 - \Sigma_{i=1}^{\infty} 1/q_i = \mu(\partial A)$ when μ is normalized Haar measure on G.

Proposition 2. *Let x be a regular Toeplitz sequence. If x is a characteristic sequence of $A \in \mathscr{C}(G)$, then $\mu(\partial A) = 0$.*

Proof. There exist arithmetic progressions $\{T_i\}_{i=1}^{\infty}$ for x such that $\Sigma_{i=1}^{\infty} 1/q_i = 1$ where as usual $T_i = \{r_i + kq_i : k \in Z\}$. It suffices to show that for all $J \in Z$

$$\sum_{i=1}^{J} 1/q_i < \mu(\mathrm{Int}\, A) + \mu(G \setminus A).$$

The progressions T_i with $i \leqq J$ can all be written as unions of progressions with modulus the least common multiple of $\{q_1, \cdots, q_J\}$ which we denote by m. So we have $\cup_{i \leq J} T_i = \cup_{i \leq \rho} T'_i$ where $T'_i = \{r'_r + km : k \in Z\}$. Let

$$U = \mathrm{Cl}(\{kmg : k \in Z\}).$$

Then U is an open closed subgroup of G of index M which divides m. If $m = dM$, then we have at least ρ/d distinct cosets $U + p_A(x) + r'_i g$. Therefore,

$$\sum_{i \leq J} 1/q_i = \rho(1/m) = (\rho/d)(1/M) < \mu\,(\mathrm{Int}\, A) + \mu(G \setminus A).$$

COROLLARY. *The regular Toeplitz sequences which are not periodic are precisely the characteristic sequences determined by some $A \in \mathscr{C}(G)$ such that G is zero-dimensional and $\mu(\partial A) = 0$.*

It is true in general that $\mu(\partial A) = 0$ implies that (X_A, σ) is uniquely ergodic and p_A is an isomorphism in the sense of ergodic theory. Thus the following is also a consequence of Proposition 2:

COROLLARY (Jacobs and Keane [5]). *If x is a regular Toeplitz sequence, then $(\mathrm{Cl}(\mathcal{O}(x)), \sigma)$ is uniquely ergodic and has discrete spectrum.*

We want to investigate regular Toeplitz sequences whose orbit closures have dynamical properties similar to substitution minimal sets. Our approach is to identify and use the properties of A, an element of $\mathscr{C}(G)$, which guarantee this. The first condition we impose is that G be an n-adic group. The rationale for this is a theorem of Gottschalk [4, theor. 3.49]. We conclude this section with a few remarks about these groups.

For an integer $n \geqq 2$ let $Z(n)$ denote the additive group of n-adic integers; that is

$$Z(n) = \left\{ \sum_{i=0}^{\infty} z_i n^i : z_i = 0, 1, \cdots, n-1 \right\}.$$

Let $U_k = \{z \in Z(n) : z_i = 0 \text{ for } i \leqq k\}$. With the U_k as a neighborhood base at 0, $Z(n)$ is a compact metric topological group. Moreover, each U_k is an open and closed subgroup of $Z(n)$ of index n^{k+1}. As usual denote the integers mod m by Z_m. There is a canonical homomorphism γ_N of $Z(n)$ onto Z_{n^N} given by $\gamma_N(\sum_{i=0}^{\infty} z_i n^i) = \sum_{i=0}^{N-1} z_i n^i \pmod{n^N}$. Let $\hat{1}_n = \sum_{i=0}^{\infty} z_i n^i$ where $z_0 = 1$ and $z_i = 0$ for $i \geqq 1$. When there is no danger of confusion we will simply write $\hat{1}$. It is easy to check that $m \to \gamma_N(m\,\hat{1}_n)$ is the canonical homomorphism of Z onto Z_{n^N} and that $\hat{1}_n$ is a generator of $Z(n)$. It can be shown that there is a homomorphism of $(Z(n), \hat{1}_n)$ onto $(Z(m), \hat{1}_m)$ if and only if every prime dividing m also divides n.

In particular, $(Z(n), \hat{1}_n)$ and $(Z(M), \hat{1}_m)$ are isomorphic if and only if they have the same prime factors.

Deterministic sequences

Let A be a closed subset of $Z(n)$ for which there exists a proper subset J of $\{0, \cdots, n-1\}$ such that the following conditions are satisfied:

1) If $z = \sum_{i=0}^{\infty} z_i n^i \in Z(n)$ and $z_i \in J$ for $i < N$ and $z_N \notin J$, then either $z + U_N \subset A$ or $z + U_N \subset Z(n) \setminus A$.

2) If $z = \sum_{i=0}^{\infty} z_i n^i \in Z(n)$ and $z_i \in J$ for $i \leqq N$, then $z + U_N$ meets both Int A and $Z(n) \setminus A$.

LEMMA 1. *If A satisfies the above, then* $\partial A = \{\sum_{i=0}^{\infty} z_i n^i : z_i \in J$ for all $i\}$.

PROOF. Since $Z(n) = \operatorname{Int} A \cup \partial A \cup (Z(n) \setminus A)$, it suffices to prove that $\operatorname{Int} A \cup (Z(n) \setminus A) = \{z = \sum_{i=0}^{\infty} z_i n^i : z_i \notin J$ for some $i\}$. It is clear from 1) that the right side is contained in the left. Let $z = \sum_{i=0}^{\infty} z_i n^i \in \operatorname{Int} A$. For some N, $z + U_N \subset A$ which implies that $z_i \notin J$ for some $i \leqq N$ by 2). Likewise if $z \in Z(n) \setminus A$.

LEMMA 2. *Let A satisfy the above and let $z' = n - 1 + \sum_{i=1}^{\infty} (n-2) n^i$. If $0 \notin J$, then $A + z'$ satisfies* 1) *and* 2) *with* $J' = \{j - 1 : j \in J\}$.

PROOF. Compute $z' + z + U_N$ when $z = \sum_{i=0}^{\infty} z_i n^i$ with $z_i \in J$ for $i < N$.

PROPOSITION 3. *Let A satisfy* 1) *and* 2) *with respect to J. Then $A \in \mathscr{C}(Z(n))$.*

PROOF. By Lemma 1 $\partial A \neq \varnothing$ and by Lemma 2 we can assume that $0 \in J$. It is clear from 2) and Lemma 1 that $\operatorname{Cl}(\operatorname{Int} A) = A$. Suppose $A + z = A$. Then $\partial A + z = \partial A$ and by Coven's Lemma $z = m\hat{1}$ [1, p. 131]. Now we can assume that $z = \sum_{i=0}^{N} z_i n^i + n^{N+1}$ which implies that $j + 1 \in J$ if $j \in J$ and $j \neq n - 1$. This brings us to the contradiction $J = \{0, \cdots, n-1\}$.

Thus conditions 1) and 2) determine a subclass of $\mathscr{C}(Z(n))$ which we will call *deterministic* and denote by $\mathscr{D}_n$. Given a complete description of A in $\mathscr{D}_n$ and the coefficients of $z \in T_A$, it is possible, as we shall see at the end of this section, to determine by direct computation as much of the sequence $p_A^{-1}(z)$ as we want. This property and condition 1) suggest the name.

Let $n = 5$, $J = \{0, 1, 2,\}$, $z = \sum_{i=0}^{\infty} z_i n^i$ where $z_i \in J$ for $i < N$ and $z_N \notin J$. The following are examples of sets in $\mathscr{D}_5$.

EXAMPLE 1: Let $W = \cup\{z + U_N : \sum_{i=0}^{N} z_i n^i$ is an even integer, $z_i \in J$ for $i < N$, $z_N \notin J\}$ and then set $A = \operatorname{Cl}(W)$. In other words, A is a closed set such that $z + U_N$ is contained in A or $Z(5)\diagdown A$ according as $\Sigma_{i=0}^{N} z_i n^i$ is even or odd if $z_i \in J$ for $i < N$ and $z_N \notin J$. Because 3 and 4 are in $\{0,1,2,3,4\}\diagdown J$ it is easy to check that 2) holds. Thus $A \in \mathscr{D}_5$.

EXAMPLE 2: This time define A so that $z + U_N$ is contained in A or $Z(5)\diagdown A$ according as z_{N-1} is even or odd with the usual conditions on z.

PROPOSITION 4. *Let $A, B \in \mathscr{D}_n$ and suppose that $0 \in J_A \cap J_B$. If (X_A, σ) and (X_B, σ) are isomorphic, then $J_A = J_B$.*

PROOF. There exists $z \in Z(n)$ such that $T_A + z = T_B$. Letting $E_A = Z(n)\backslash T_A = \partial A + \hat{1}$ we also have $E_A + z = E_B$. If we can show that $z = m\,\hat{1}$, then it follows that $J_A = J_B$.

Assume $z \notin Z\hat{1}$ and pick q in $\{0,\cdots,n-2\}$ such that q appears infinitely often in z. Then following Coven [1, pp. 131–132] we see that $[q + j] \in J_B$ for all $l \in J_A$ where $[\cdot]$ denotes the residue mod n. Applying the same reasoning to $E_B - z = E_A$ we see that $[n - q - 1 + j'] \in J_A$ for all $j' \in J_B$. Combining these we have $[[q + j] + n - q - 1] \in J_A$ for all $j \in J_A$ or j in J_A implies $[(n-1) + j] \in J_A$ which yields the contradiction $J_A = \{0,\cdots,n-1\}$.

In view of Lemma 2 and Proposition 4 we now make the standing assumption that $0 \in J_A$ whenever $A \in \mathscr{D}_n$.

Given z in $Z(n)$ and $N > 0$ there is a certain interval of integers m such that $z + m\,\hat{1}$ and z have the same coefficients for $i \geqq N$. These intervals will play a basic role in our theory. Their relevance for substitutions was discovered by Coven and Keane [2, theor.1, p. 95]. In the next few results we will describe their properties.

PROPOSITION 5. *Let $z = \Sigma_{i=0}^{\infty} z_i n^i$ and let $N \geqq 1$. Set $a_N(z) = -(1 + \Sigma_{i=0}^{N-1} z_i n^i)$ and $b_N(z) = n^N - \Sigma_{i=0}^{N-1} z_i n^i$. Then as m ranges through the integers in $(a_N(z), b_N(z))$ $z + m\,\hat{1} = \Sigma_{i=0}^{\infty} w_i n^i$ has the following properties:*

a) $w_i = z_i$ *for* $i \geqq N$

b) $\Sigma_{i=0}^{N-1} w_i n^i$ *takes on all integer values in* $(-1, n^N)$.

PROOF. If we establish a) and check that $(a_N(z), b_N(z))$ contains n^N integers, then b) follows because $\Sigma_{i=0}^{N-1} w_i n^i$ would assume n^N different values $(z + m\,\hat{1} = z + m'\hat{1}$ if and only if $m = m')$. To prove a) we need the following lemma:

LEMMA 3. *Let $b = \Sigma_{i=0}^{N} b_i n^i$ be a positive integer where $0 \leqq b_i < n$. Then $0 < m < b$ if and only if $m = \Sigma_{i=0}^{N} m_i n^i$ where the m_i satisfy the following conditions:*

a) $0 \leqq m_i < n$

b) *some* $m_i \neq 0$

c) *there exists* $j \leqq N$ *such that* $m_j < b_j$ *and* $m_i = b_i$ *for* $j < i \leqq N$.

PROOF. Assume $0 < m < b$. There is an expression for m such that a) and b) hold. Because $m_i n^i \leqq m < b < n^{N+1}$ it contains at most N terms. Suppose $m_N > b_N$. Then $b \leqq n^N - 1 + b_N n^N < m_N n^N \leqq m$. If $m_i = b_i$ for $j < i \leqq N$, then $\Sigma_{i=0}^{j} m_i n^i \leqq \Sigma_{i=0}^{j} b_i n^i$ and we can repeat the argument on m_j.

For the converse note that without loss of generality we can assume that $m_N < b_N$. Then $m \leqq n^N - 1 + m_N n^N = (m_N + 1)n^N - 1 < b_N m^N \leqq b$.

PROOF OF a). First we consider m in $(0, b_N(z))$ and we rewrite $b_N(z)$:

$$b_N(z) = n^N - \sum_{i=0}^{N-1} z_i n^i = \sum_{i=0}^{N-1} (n-1)n^i + 1 - \sum_{i=0}^{N-1} z_i n^i$$
$$= (n - z_0) + \sum_{i=1}^{N-1} (n - z_i - 1)n^i.$$

By Lemma 3, $m = \Sigma_{i=0}^{N-1} m_i n^i$, $0 \leqq m_i < n$ such that $m_i = n - z_i - 1$ for $j < i < N$ & $m_j < n - z_j - 1$. Therefore, the jth coordinate of $z + m\hat{1}$ is at most $m_j + z_j + 1 < n$ and the ith coordinate for $j < i < N$ is $M_i + z_i = n - 1 < n$. The $N - 1$st coordinate of $z + m\hat{1}$ is at least $m_{N-1} + z_{N-1}$ and the Nth coordinate of $z + m\hat{1}$ is z_N. Therefore, a) holds for $0 < m < b_N$.

For $a_N < m < 0$ apply the above to $-z = n - z_0 + \Sigma_{i=1}^{\infty}(n - 1 - z_i)n^i$ and $0 < m < b_N(-z) = 1 + \Sigma_{i=0}^{N-1} z_i n^i = -a_N(z)$.

REMARK.

a) If $z_i \neq 0$ for infinitely many i, then $a_N(z)$ decreases to $-\infty$.

b) If $z_i \neq n - 1$ for infinitely many i, then $b_N(z)$ increases to ∞.

COROLLARY. *Let* $z, w \in Z(n)$. *If* $z_i = w_i$ *for all* $i \geqq N \geqq 1$, *then there exists* m *in* $(a_N(z), b_N(z))$ *such that* $w = z + m\hat{1}$.

PROOF. There are at most n^N elements like w given z. By Proposition 5, $z + m\hat{1}$ with $m \in (a_N(z), b_N(z))$ produces n^N of them.

COROLLARY. *Let* $z \in Z(n)$. *If* $m \in (a_N(z), b_N(z))$, *then* $(a_N(z + m\hat{1}), b_N(z + m\hat{1})) = (a_N(z) - m, b_N(z) - m)$.

PROOF. Using the homomorphism γ_N it is easy to see that $b_N(z + m\hat{1})$ and $b_N(z) - m$ are congruent mod n^N, and then note that 0 is in both intervals.

Suppose $A \in \mathscr{D}_n$, $0 \in J$, and $z \in T_A$. Then to compute $x = p_A^{-1}(z)$ on $[-M, M]$ one proceeds as follows:

I) Find N such that $a_N(z) < -M < 0 < M < b_N(z)$.

II) Find m such that $z_m \notin J$ and $m \geqq N$.

III) Determine whether $\Sigma_{i=0}^{\rho} w_i n^i + U_\rho$ is contained in A or $Z(n) \setminus A$ when $\rho \leqq m$, $w_\rho \notin J$ and $w_i \in J$ for $i < \rho$.

IV) For q in $(a_m(z), b_m(z))$, $x(q) = \chi_A(\Sigma_{i=0}^{m} z_i n^i + q\hat{1})$ and this data was determined in III).

Let m and n have the same prime factors. Then there is a unique isomorphism ψ of $(Z(n), \hat{1}_n)$ onto $(Z(m), \hat{1}_m)$ such that $\psi(\hat{1}_n) = \hat{1}_m$. Moreover, ψ is an isomorphism of topological groups.

PROPOSITION 6. *Let $A \in \mathcal{D}_n$ with $J_A \neq \{0\}$. If $\psi(A) \in \mathcal{D}_m$, then there exist p and q such that $n^p = m^q$.*

PROOF. In the next section we will prove a more delicate theorem and point out how the techniques can be applied to this proposition.

Substitution-like minimal sets

In this section, which is the heart of the paper, we will define substitution-like minimal sets, consider the isomorphism problem within this class, and determine their factors.

Let $A \in \mathcal{D}_n$. We say A is substitution-like provided that given any $N > 0$ and any two points $z = \Sigma_{i=0}^{\infty} z_i n^i$ and $z' = \Sigma_{i=0}^{\infty} z'_i n^i$ in T_A such that $z_i = z'_i \in J$ for $i < N$, then $\chi_A(z + m\hat{1}) = \chi_A(z' + m\hat{1})$ for all m in $(a_N(z), b_N(z))$ if $\chi_A(z) = \chi_A(z')$. The collection of substitution-like elements of $\mathcal{D}_n$ will be denoted by $\mathcal{S}\ell_n$, and if $A \in \mathcal{S}\ell_n$ for some $n \geqq 2$, then (X_A, σ) is a substitution-like minimal set. Example 1 is substitution-like and Example 2 is not.

We denote the cardinality of a finite set F by $|F|$.

PROPOSITION 7. *Let $A \in \mathcal{S}\ell_n$. If $n - 1 \notin J$, then $|p_A^{-1}(z)| = 2$ for all $z \in \partial A$. If $n - 1 \in J$, then $|p_A^{-1}(z)| = 2$ for all $z \in \partial A \setminus Z\hat{1}$ and $3 \leqq |p_A^{-1}(0)| \leqq 4$.*

PROOF. Let $x, y \in p_A^{-1}(z)$ such that $x(0) = y(0)$. There exist sequences of almost automorphic points $\{x_k\}$ and $\{y_k\}$ converging to x and y. Let $z = \Sigma_{i=0}^{\infty} z_i n^i$, $p_A(x_k) = \xi_k = \Sigma_{i=0}^{\infty} \xi_{ki} n^i$, and $p_A(y_k) = \zeta_k = \Sigma_{i=0}^{\infty} \zeta_{ki} n^i$. Given $N > 0$ pick k so that $x_k = x$ and $y_k = y$ on $(a_N(z), b_N(z))$, and $\xi_{ki} = z_i = \zeta_{ki}$ for $0 \leqq i \leqq N$. On $(a_N(z), b_N(z))$ we now have $x(m) = \chi_A(\xi_k + m\hat{1})$ and $y(m) = \chi_A(\zeta_k + m\hat{1})$. Because A is substitution-like and $x(0) = y(0)$ it follows that $x = y$ on $(a_N(z), b_N(z))$. If $z \notin Z\hat{1}$, then $a_N(z) \to -\infty$ and $b_N(z) \to \infty$ as $N \to \infty$ and $x = y$ independent of where $n - 1$ is. When $z \in Z\hat{1}$ we may as well assume that $z = 0$.

Since $b_N(0) \to \infty$, $x(m) = y(m)$ for $m \geqq 0$. For $x, y \in p_A^{-1}(0)$ with $x(-1) = y(-1)$ we have $x(m) = y(m)$ for $m < 0$ because $a_N(-\hat{1}) \to -\infty$.

We know that there exist x and y in $p_A^{-1}(z)$ such that $x(0) \neq y(0)$. By the above it follows that $p_A^{-1}(z) = \{x, y\}$ unless $z \in Z\hat{1}$ and $n-1 \in J$. If $z = 0$ and $n - 1 \in J$, then it is clear that $|p_A^{-1}(0)| \leqq 4$. Let q be the smallest integer in $\{0, \cdots, n-1\}$ which is not in J. If $n - 1 \in J$, then there exist $\xi_k = qn^k + \Sigma_{i=k+1}^{\infty} \xi_{ki} n^i$ and $\zeta_k = qn^k + \Sigma_{i=k+1}^{\infty} \zeta_{ki} n^i$ in T_A such that

$$\chi_A(\xi_k - \hat{1}) \neq \chi_A(\zeta_k - \hat{1}).$$

It follows that there exist $x, y \in p_A^{-1}(0)$ such that $x(0) = y(0)$ and $x(-1) \neq y(-1)$. Therefore, $3 \leqq |p_A^{-1}(0)|$ if $n - 1 \in J$, which completes the proof.

COROLLARY. *Let $x, y \in p_A^{-1}(z)$ when $z \notin T_A$. Then $x = y$ if and only if $x(m) = y(m)$ for some m such that $z + m\hat{1} \in \partial A$ provided that either $z \notin Z\hat{1}$ or $n - 1 \notin J$.*

It is easy to check that the following are true for $A \in \mathcal{S}\ell_n$:

a) If $n - 1 \notin J$, then $p_A^{-1}(0)$ consists of a pair of negatively asymptotic points x and y such that $x(m) = y(m)$ for all $m < 0$.

b) If $n - 1 \notin J$, then $p_A^{-1}(\mu)$ consists of a pair of positively asymptotic points x and y such that $x(m) = y(m)$ for all $m > 0$. $\mu = \Sigma_{i=0}^{\infty} j_M z^i$ and $j_M = \max\{j \in J\}$.

c) If $n - 1 \in J$, then $p_A^{-1}(0)$ contains at least one pair of positively asymptotic points and at least one pair of negatively asymptotic points.

d) Let $x, y \in p_A^{-1}(0)$. If $n - 1 \in J$, then the following are equivalent:

i) x and y are positively (negatively) asymptotic
ii) $x(m) = y(m)$ for some $m \geqq 0$ $(m \leqq -1)$ such that $m\hat{1} \in \partial A$
iii) $x(0) = y(0)$ $\{x(-1) = y(-1)\}$
iv) $x(m) = y(m)$ for all $m \geqq 0$ $(m \leqq -1)$.

e) Let $x, y \in p_A^{-1}(z)$ where $z \notin T_A$. If x and y are positively (negatively) asymptotic, then $z = \mu + m\hat{1}$ $(z = m\hat{1})$ for some m in Z.

Let A and B be elements of $\mathcal{S}\ell_n$ with $J_A = J_B = J$. We say A and B are semi-dual if there exists $N > 0$ such that for each $z = \Sigma_{i=0}^{\infty} z_i n^i$ with $z_i \in J$ for $i \leqq N$ we have one of the following:

$$A \cap (z + U_N) = B \cap (z + U_N)$$

$$A \cap (z + U_N) = B_0 \cap (z + U_N)$$

where $B_0 = \mathrm{Cl}(Z(n) \setminus B)$.

THEOREM 1. *Let $A, B \in \mathcal{S}\ell_n$. Then (X_A, σ) and (X_B, σ) are isomorphic if and only if $J_A = J_B$ and A and B are semi-dual.*

PROOF. Suppose ϕ is an isomorphism of (X_A, σ) onto (X_B, σ). From e) it follows that $p_B(\phi(x)) = m\hat{1}$ if $x \in p_A^{-1}(0)$. Replacing ϕ by $\sigma^{-m} \cdot \phi$ we can assume that $p_B \cdot \phi = p_A$. From Proposition 4 we know that $J_A = J_B = J$. Assume A and B are not semi-dual. Then there exists a sequence of integers N_k going to infinity and a sequence ζ_k in $Z(n)$ converging to $\zeta \in \partial A$ such that $\zeta_{ki} \in J$ for $i \leqq N_k$ and $\zeta_k + U_{N_k}$ does not satisfy either of the semi-dual equations. There are exactly four ways that this can happen, and so without loss of generality we can assume it happens the same way for all k. We will argue one of the four cases, since they are all essentially the same.

We will assume that there exist z_k and w_k in $A \cap (\zeta_k + U_{N_k})$ such that $z_k \in B \cap (\zeta_k + U_{N_k})$ and $w_k \notin B \cap (\zeta_k + U_{N_k})$. Clearly $\{z_k\}$ and $\{w_k\}$ converge to ζ as $k \to \infty$. Because $\partial A = \partial B$ we can assume $z_k, w_k \in T_A = T_B$ for all k. Letting $x_k = p_A^{-1}(z_k)$ and $y_k = p_A^{-1}(w_k)$ we can assume that $x_k \to x$ and $y_k \to y$ as $k \to \infty$. Note that $x(0) = 1 = y(0)$ and $\phi(x)(0) \neq \phi(y)(0)$. These equations have contradictory implications; the specific contradictions are the following:

i) If $\zeta \notin Z\hat{1}$ or $n - 1 \notin J$, then $x = y$ and $\phi(x) \neq \phi(y)$.

ii) If $n - 1 \in J$ and $\zeta = m\hat{1}$ with $m \geqq 0$, then x and y are positively asymptotic and $\phi(x)$ and $\phi(y)$ are not positively asymptotic.

iii) If $n - 1 \in J$ and $\zeta = m\hat{1}$ with $n < 0$ then x and y are negatively asymptotic and $\phi(x)$ and $\phi(y)$ are not negatively asymptotic.

For the converse we will show that the canonical map of the characteristic sequences of X_A onto the characteristic sequences of X_B is uniformly continuous. Let $M > 0$ be given. There exists M_0 such that if $x = y$ on $(-M_0, M_0)$, then $d(p_A(x), p_A(y)) < \gamma = \inf\{d(z, U_N): z \notin U_N\}$ where d is an invariant metric on $Z(n)$ and N is given by the definition of semi-dual. Note that $d(z, z') < \gamma$ implies $z - z' \in U_N$. Let $M' = M + M_0$. It is easy to check that if x and y are characteristic sequences in X_A and they agree on $(-M', M')$, then $p_B^{-1}(p_A(x))$ and $p_B^{-1}(p_A(y))$ agree on $(-M, M)$.

THEOREM 2. *Let A be substitution-like. If $n - 1 \notin J$, then (X_A, σ) is a prime extension* (see [8, sect. 6]) *of $(Z(n), \hat{1})$.*

PROOF. Let R be a closed invariant equivalence relation on X_A and let P be the proximal relation on X_A, which equals the equicontinuous structure relation because (X_A, σ) is locally almost periodic. It suffices to show that $P \subset R$. We will show that $R \neq \Delta$ implies $R \cap P \neq \Delta$ and $\Delta \neq R \subset P$ implies $R = P$.

Suppose that $R \neq \Delta$ and $R \cap P = \Delta$. Let (Y, ϕ) denote $(X_A/R, \sigma/R)$, θ the canonical homomorphism of (X_A, σ) onto (Y, ϕ), and $\hat{\theta}$ the induced map of $(Z(n), \hat{1})$ onto (G, g) — the maximal equicontinuous factor of (Y, ϕ). Without loss of generality $\hat{\theta}$ is a group homomorphism. Let H = kernel of $\hat{\theta}$. If $H = \{0\}$, then $R \subset P$ and $R = R \cap P = \Delta$; thus $H \neq \{0\}$. Because $R \cap P = \Delta$ it is easy to see that $T_A + H = T_A$. It then follows that $H \subset Z\hat{1}$ [1, p. 131] and then $H = \{0\}$ because H is closed. We have shown that $\Delta \neq R$ implies $R \cap P \neq \Delta$.

To show that $\Delta \neq R \subset P$ implies $R = P$ it will suffice to prove the following lemma:

LEMMA 4. *Let z and $z' \in \partial A$ such that $z - z' \notin Z\hat{1}$, let $x, y \in p_A^{-1}(z)$ with $x(0) \neq y(0)$, and let $x', y' \in p_A^{-1}(z')$ with $x'(0) \neq y'(0)$. If $n - 1 \notin J$, then there exists a sequence $\{m_k\}$ such that $(\sigma^{m_k}(x), \sigma^{m_k}(y))$ converges to either (x', y') or (y', x').*

PROOF. For each $N > 0$ there exist ξ_N and ζ_N in T_A such that $\xi_{Ni} = z_i = \zeta_{Ni}$ for all $i < N$ and that $x_N = p_A^{-1}(\xi_N)$ and $y_N = p_A^{-1}(\zeta_N)$ equal x and y on $(a_N(z), b_N(z))$. Furthermore, there exists m_N in $(a_N(z), b_N(z))$ such that $\xi'_{Ni} = z'_i = \zeta'_{Ni}$ for $i < N$ where $\xi'_N = \xi_N + m_N\hat{1}$. Because $A \in \mathcal{S}\ell_n$ and $(a_N(z'), b_N(z')) = (a_N(z) - m_N, b_N(z) - m_N)$ we have the following equivalences: $\chi_A(\xi_N) = \chi_A(\zeta_N)$ if and only if $\chi_A(\xi_N + m\hat{1}) = \chi_A(\zeta_N + m\hat{1})$ on $(a_N(z), b_N(z))$ if and only if $\chi_A(\xi'_N + m\hat{1}) = \chi_A(\zeta'_N + m\hat{1})$ on $(a_N(z'), b_N(z'))$ if and only if $\chi_A(\xi'_N) = \chi_A(\zeta'_N)$. Thus $\chi_A(\xi'_N) \neq \chi_A(\zeta'_N)$. On the interval $(a_N(z'), b_N(z'))$ we now have $\sigma^{m_N}(x)(m) = x(m + m_N) = \chi_A(\xi_N + (m_N + m)\hat{1}) = \chi_A(\xi'_N + m\hat{1}) = x'(m)$ (renaming if necessary). A similar equation holds for y' because $p_A^{-1}(z') = \{x', y'\}$ and $\chi_A(\xi'_N) \neq \chi_A(\zeta'_N)$. If follows that $\sigma^{m_N}(x) \to x'$ and $\sigma^{m_N}(y) \to y'$ as $N \to \infty$ to complete the proof.

Note that we only used the hypothesis $n - 1 \notin J$ to assert that $p_A^{-1}(z') = \{x', y'\}$ in the proof of Lemma 4 and that R plays no role in Lemma 4. Suppose $n - 1 \in J$. Then we still have $R \neq \Delta$ implies $R \cap P \neq \Delta$. If $\Delta \neq R \subset P$, then $P[x] = R[x]$ when $p_A(x) \notin Z\hat{1}$ and there are at most two equivalence classes of R in $p_A^{-1}(0)$. Moreover, it is easy to check that in the quotient proximal implies doubly asymptotic. Therefore, if $n - 1 \in J$, then either (X_A, σ) is a prime extension of $(Z(n), \hat{1})$ or a prime extension of a prime extension of $(Z(n), \hat{1})$.

PROPOSITION 8. *If $A \in \mathcal{D}_n$ and $J = \{0\}$, then $A \in \mathcal{S}\ell_n$.*

PROOF. Let z and $z' \in T_A$ be such that $z_i = z'_i \in J$ for $i < N$. Hence $z_i = 0$ for $i < N$, and $(a_N(z), b_N(z)) = (-1, n^N)$. For $0 < m < n^N$, $z + m\hat{1}$ and $z' + m\hat{1}$

have the same coefficients for $i < N$ at least one of which is non-zero. Then $\chi_A(z + m\hat{1}) = \chi_A(z' + m\hat{1})$ for $0 < m < n^N$.

THEOREM 3. *Let A and B be elements of $\mathcal{S}\ell_n$ and $\mathcal{S}\ell_m$ such that $\partial A \neq \{0\}$. If (X_A, σ) and (X_B, σ) are isomorphic, then there exist positive integers p and q such that $n^p = m^q$.*

PROOF. Let ϕ be an isomorphism of (X_A, σ) onto (X_B, σ). Clearly $\partial B \neq \{0\}$. Because $p_A^{-1}(0)$ and $p_B^{-1}(0)$ have special dynamical properties we can assume that $\phi(p_A^{-1}(0)) = p_B^{-1}(0)$ by replacing ϕ by $\sigma^k \cdot \phi$ for some suitable k. It follows that the isomorphism of $(Z(n), \hat{1}_n)$ onto $(Z(m), \hat{1}_m)$ induced by ϕ is the group isomorphism ψ. Therefore n and m have the same prime divisors.

The next lemma will show that near 0, $\partial A = \partial B$. In proving this lemma we will make use of the standard metric for Ω given by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1/(q + 1)$ for $x \neq y$ where $q = \min\{|i| : x(i) \neq y(i)\}$.

LEMMA 5. *There exists I such that $z = \sum_{i=I+1}^{\infty} z_i n^i \in \partial A$ implies $\psi(z) \in \partial B$.*

PROOF. Pick I such that $d(\phi(x), \phi(y)) < 1/4$ whenever $x = y$ on $(-n^I, n^I)$ and such that $p_A(x) \in T_A \cap U_I$ implies $x(k) = x'(k)$, $k = -1, 0$, for some $x' \in p_A^{-1}(0)$. Let $z = \sum_{i=l}^{\infty} z_i n^i \in \partial A$ with $z_l \neq 0$ and $l > I$, and let $x, y \in p_A^{-1}(z)$ such that $x(0) \neq y(0)$. It suffices to show that $\phi(x)(0) \neq \phi(y)(0)$.

There exist ζ and ζ' in $Z(n)$ such that $\zeta_i = z_i = \zeta'_i$ for $i \leqq l$; $\zeta, \zeta' \in T_A \cap U_I$; and $\chi_A(\zeta + m\hat{1}_n) = x(m)$ and $\chi_A(\zeta' + m\hat{1}_n) = y(m)$ on $(a_N(z), b_N(z))$ with $N = l + 1$. It is easy to check that $a_N(z) < -n^I$, $b_N(z) > n^I$, $b_l(\zeta) > n^I$, and $a_l(\zeta - \hat{1}_n) < -n^I$. Therefore, there exist x' and y' in $p_A^{-1}(0)$ such that $x = x'$ and $y = y'$ on $(-n^I, n^I)$. Now $x'(0) \neq y'(0)$ implies that x' and y' and hence $\phi(x')$ and $\phi(y')$ are not positively asymptotic. Thus $\phi(x')(0) \neq \phi(y')(0)$. We now have $1 = d(\phi(x'), \phi(y')) \leqq d(\phi(x'), \phi(x)) + d(\phi(x), \phi(y)) + d(\phi(y), \phi(y')) < 1/4 + d(\phi(x), \phi(y)) + 1/4$, which implies that $d(\phi(x), \phi(y)) > 1/2$ and hence 1. It follows that $\phi(x)(0) \neq \phi(y)(0)$ and the proof of the lemma is completed.

COROLLARY. *There exists an open and closed neighborhood V of the identity in $Z(n)$ such that $\psi(\partial A \cap V) = \partial B \cap \psi(V)$.*

LEMMA 6. *Let m and n be positive integers with the same prime factors, let j be a positive integer less than m, and for each positive integer i let*

$$jm^i = \sum_{l=\lambda(i)}^{\sigma(i)} k_l n^l$$

where $0 \leqq k_l \leqq n-1$, $k_{\lambda(i)} \neq 0$, and $k_{\sigma(i)} \neq 0$. *If there are no positive integers p and q such that* $n^p = m^q$, *then*

$$\lim_{i \to \infty} \sigma(i) - \lambda(i) = \infty.$$

PROOF. Suppose there exists an increasing sequence of positive integers $\{i_\gamma\}_{\gamma=1}^{\infty}$ such that $\sigma(i_\gamma) - \lambda(i_\gamma) \leqq M$ for some M. Then $jm^i \leqq n^{\lambda(i)} \Sigma_{l=0}^{M} (n-1)n^l = n^{\lambda(i)}(n^M - 1)$ if $i = i_\gamma$ for some γ. Let $n = p_1^{e_1} \cdots p_s^{e_s}$ and $m = p_1^{f_1} \cdots p_s^{f_s}$ where the p_i are primes and the e_i and f_i are all positive. Since $n^{\lambda(i)} \leqq k_{\lambda(i)} n^{\lambda(i)} \leqq jm^i$, for $i = i_\gamma$ we have

$$1 \leqq \frac{jm^i}{n^{\lambda(i)}} = jp_1^{if_1 - \lambda(i)e_1} \cdots p_s^{if_s - \lambda(i)e_s} \leqq n^M - 1.$$

Because $n^{\lambda(i)}$ divides jm^i there exists P such that for all i, $P \leqq if_t - \lambda(i)e_t$ when $1 \leqq t \leqq s$. It follows that there exists Q such that for $1 \leqq t \leqq s$ and for all i_γ we have

$$P \leqq i_\gamma f_t - \lambda(i_\gamma)e_t \leqq Q.$$

By dividing by i_γ and letting γ go to infinity we find that for $1 \leqq t \leqq s$

$$\frac{f_t}{e_t} = \lim_{\gamma \to \infty} \frac{\lambda(i_\gamma)}{i_\gamma},$$

which implies that there exist positive integers p and q such that $n^p = m^q$.

We can now complete the proof of the theorem. Assume that there do not exist positive integers p and q such that $n^p = m^q$. Let V be given by the Corollary of Lemma 5. There exists N such that $z = \Sigma_{i=N}^{\infty} z_i n^i$ implies $z \in V$ and $w = \Sigma_{i=N}^{\infty} w_i m^i$ implies $w \in \psi(V)$. Pick $j' = 0$ in $J_B \subset \{0, \cdots, m-1\}$. As in Lemma 6 write

$$j'm^i = \sum_{l=\lambda(i)}^{\sigma(i)} k_l n^l.$$

Note that $\lambda(i) \to \infty$ as $i \to \infty$ because in $Z(m)$ $\lim_{i \to \infty} j'm^i = 0$ and because $\hat{\phi}(\hat{1}_n) = \hat{1}_m$. Choose i so that $i > N$, $\lambda(i) > N$, and $\sigma(i) - \lambda(i) > 3$.

Consider $z = j_0 n^{\lambda(i)} + j_1 n^{\lambda(i)+1}$ with $j_0, j_1 \in J_A$. Since $z \in \partial A \cap V$, $z = \Sigma_{i=0}^{M} w_i m^i$ with $w_i \in J_B$ for all i. Observe that $j_0 n^{\lambda(i)} + j_1 n^{\lambda(i)+1} \leqq (n-1)n^{\lambda(i)} + (n-1)n^{\lambda(i)+1} = n^{\lambda(i)+2} - n^{\lambda(i)} < n^{\lambda(i)+2} < n^{\sigma(i)-1} < n^{\sigma(i)} < j'm^i$. Without loss of generality we can assume that $m < n$ and then $n^{\sigma(i)} < j'm^i < m^{i+1}$ implies that $n^{\sigma(i)-1} < m^i$. It follows that $M < i$. Consequently $w =$

$\Sigma_{i=0}^{M} w_i m^i + j' m^i \in \partial B \cap \psi(V) = \psi(\partial A \cap V)$ and $\zeta = \psi^{-1}(w) \in \partial A \cap V$. By choosing z in various ways we will obtain the contradiction $J_A = \{0, \cdots, n-1\}$.

First let $j_0 = k_{\lambda(i)}$ and $j_1 = 0$. This forces all multiples of $k_{\lambda(i)}$ taken mod n to be in J_A. In particular, $n - k_{\lambda(i)} \in J_A$. Similarly, if $k_{\lambda(i)+1} \neq 0$, then $n - k_{\lambda(i)+1} \in J_A$ by taking $j_0 = 0$ and $j_1 = k_{\lambda(i)+1}$. Next let $j_0 = n - k_{\lambda(i)}$ and $j_1 = 0$ or $n - k_{\lambda(i)+1}$ according as $k_{\lambda(i)+1}$ is or is not zero. It follows that $1 \in J_A$. Finally letting $j_0 = 1$ and $j_1 = 0$, we see that $J_A = \{0, \cdots, n-1\}$.

PROPOSITION 9. *Let $A \in \mathcal{S}\ell_n$ and $B \in \mathcal{S}\ell_m$ and suppose that $\partial A = \{0\}$. If (X_A, σ) and (X_B, σ) are isomorphic, then there exists N such that $U_N \cap A = U_N \cap \psi^{-1}(B)$ or $U_N \cap \mathrm{Cl}(Z(n) \setminus A) = U_N \cap \psi^{-1}(B)$ where ψ is the canonical isomorphism of $(Z(n), \hat{1}_n)$ onto $(Z(m), \hat{1}_m)$.*

PROOF. Let Ψ be an isomorphism of (X_A, σ) onto (X_B, σ). Clearly ∂B must also be $\{0\}$ and if $x \in p_A^{-1}(0)$, then $p_B \cdot \Psi(x) = k\hat{1}_m$. Replacing Ψ by $\sigma^{-k} \cdot \Psi$ yields $p_B \cdot \Psi = \psi \cdot p_A$. Now let $B' = \psi^{-1}(B)$, and notice that $X_{B'} = X_B$, $\partial B' = \{0\}$ and $T_A = T_{B'}$. The method of proof is now very similar to that of Theorem 1.

First we show that there exists N_0 such that for every coset $z + U_N \neq U_N$ with $N > N_0$ and $z + U_N \subset U_{N_0}$ either $z + U_N \subset B'$ or $z + U_N \subset Z(n) \setminus B'$. If this were false there would exist sequences $\{\xi_k\}$ and $\{\zeta_k\}$ in T_A converging to 0 such that $\{\xi_k\} \subset B'$ and $\{\zeta_k\} \subset Z(n) \setminus B'$ and such that $x_k(0) = y_k(0)$ where $x_k = p_A^{-1}(\xi_k)$ and $y_k = p_A^{-1}(\zeta_k)$. We can assume that $\{x_k\}$ and $\{y_k\}$ converge to x and y. It follows that $x, y \in p_A^{-1}(0)$ and $x(0) = y(0)$, and hence $x = y$. Since $\Psi(x_k)(0) \neq \Psi(y_k)(0)$, we have the contradiction $\Psi(x) \neq \Psi(y)$.

To finish the proof we can proceed exactly as in the first half of the proof of Theorem 1 because the above shows that B' is substitution-like in U_{N_0} and $\partial A' = \partial B' = \{0\} \subset U_{N_0}$.

We conclude this section with an outline of the proof of Proposition 6. Instead of Lemma 5 all we need is the observation that $\partial\psi(A) = \psi(\partial A)$. Since Lemma 6 is number theoretical, we have it at our disposal. Now we can use the same technique to show that $J_A = \{0, 1, \cdots, n-1\}$ with $V = Z(n)$.

Substitutions

A substitution of length $n (n \geqq 2)$ is a map θ of $\{0, 1\}$ into the n-blocks on 0 and 1. Let $\theta(0) = a_0 a_1 \cdots a_{n-1}$ and $\theta(1) = b_0 b_1 \cdots b_{n-1}$ and let $\theta(i)_r = a_r$ or b_r according as $i = 0$ or 1. The substitution θ induces a continuous map θ of Ω into Ω by $\theta(x)(m) = \theta(x(k))_r$ where $m = kn + r$ and $0 \leqq r < n$. The general problem for substitutions is to determine the dynamical properties of $\mathrm{Cl}(\mathcal{O}(x))$

where $\theta(x) = x$. In this section we will present some new results of this type which follow from our results on substitution-like minimal sets.

Let λ be the map on 2-blocks given by $\lambda(pq) = \theta(p)_{n-1}\theta(q)_0$. The number of points $x \in \Omega$ such that $\theta(x) = x$ equals the number of fixed points of λ. It can happen that λ has no fixed points; for example, $\theta(0) = 11$ and $\theta(1) = 01$. To prevent this one usually replaces θ by θ^2 [2, sect. 3], but we will not do this.

For a substitution θ of length n let $I = \{i: a_i = b_i\}$ and let $J = \{j: a_j \neq b_j\}$. We say θ is discrete provided that a) $I \neq \emptyset$, b) $J \neq \emptyset$, and c) $a_i \neq 0$ for some i and $b_i \neq 1$ for some i. Henceforth, we will assume that θ is a discrete substitution.

We will need the following formulas, which are easily established.

Lemma 7. *Let* $m = \Sigma_{k=0}^{N} m_k n^i \in Z$ *with* $0 \leqq m_k < n$.

a) $\theta^{N+1}(\alpha)_m = \theta(\cdots\theta(u(\alpha)_{m_N})_{m_{N-1}}\cdots)_{m_0}$.

b) *If* $m_k \in J$ *for all* k, *then* $\theta^{N+1}(0)_m \neq \theta^{N+1}(1)_m$.

c) *If* $m_{k'} \in I$, *then* $\theta^{N+1}(\alpha)_m$ *is independent of* m_k *for* $k > k'$.

We now associate a closed subset of $Z(n)$ with θ. Let $z = \Sigma_{i=0}^{N} z_i n^i \in Z(n)$ with $z_i \in J$ for $i < N$ and $z_N \in I$. Because z is also a positive integer $\theta^{N+1}(0)_z$ makes sense. Let

$$A_\theta = \mathrm{Cl}\left[\bigcup\left\{z + U_N : z = \sum_{i=1}^{N} z_i n^i \text{ with } z_i \in J \Leftrightarrow i < N \,\&\, \theta^{N+1}(0)_z = 1\right\}\right].$$

It is clear that with z as above $z + U_N$ is contained in A_θ or $Z(n) \setminus A_\theta$ according as $\theta^{N+1}(0)_z$ equals 1 or 0.

Proposition 10. *If* θ *is a discrete substitution of length* n, *then* $A_\theta \in \mathcal{D}_n$.

Proof. We have defined A_θ so that condition (1) in the definition of $\mathcal{D}_n$ is satisfied and so it remains to check (2). Consider $z = \Sigma_{i=1}^{N} z_i n^i$ with $z_i \in J$ for $0 \leqq i \leqq N$. First suppose we can find $i_0, i_1 \in I$ such that $a_{i_0} = 0$ and $a_{i_1} = 1$. Now let $\zeta = z + i_0 n^{N+1}$ and $\zeta' = z + i_1 n^{N+1}$. Then $\zeta + U_{N+1} \subset A_\theta$ and $\zeta' + U_{N+1} \subset Z(n) \setminus A_\theta$ or vice versa by b) of the above lemma. Since they are both subsets of $z + U_N$, (2) holds. If we can not find i_0 and i_1 as above, then there exists $j_0 \in J$ such that $b_{j_0} = 0$ and $a_{j_0} = 1$. Let $i_0 \in I$. In this case let $\zeta = z + i_0 n^{N+1}$ and $\zeta' = z + j_0 n^{N+1} + i_0 n^{N+2}$ and proceed as above.

Instead of using A_θ as a subscript for χ, T, p, and X we will simply use θ.

We have been assuming that if $A \in \mathcal{D}_n$, then $0 \in J$. We will temporarily drop this assumption until we show that X_θ is the usual minimal set associated with θ and that replacing A by $A + \gamma$ as in Lemma 2 so that $0 \in J$ corresponds to a simple rearrangement of θ.

PROPOSITION 11. *The set $p_\theta^{-1}(0)$ is invariant under θ.*

PROOF. Let $z = \Sigma_{i=0}^{\infty} z_i n^i$ and set $nz + r\hat{1} = rn^0 + \Sigma_{i=0}^{\infty} z_i n^{i+1}$ where $0 \leqq r \leqq n$. Then for $z \in T_\theta$ it is immediate that $\theta(\chi_\theta(z))_r = \chi_\theta(nz + r\hat{1})$. Let z_α be a sequence in T_θ converging to 0 such that $p_\theta^{-1}(z_\alpha)$ converges to $x \in p_\theta^{-1}(0)$. We can assume that $p_\theta^{-1}(nz_\alpha)$ converges to $y \in p_\theta^{-1}(0)$. Then for $m \in Z$ by choosing α large we have $\theta(x)(m) = \theta(x(k))_r = \theta(\chi_\theta(z_\alpha + k\hat{1}))_r = \chi_\theta(r\hat{1} + nz_\alpha + nk\hat{1}) = y(r + nk) = y(m)$ where $m = nk + r$ and $0 \leqq r < n$. Then $\theta(x) = y \in p_\theta^{-1}(0)$.

PROPOSITION 12. *If θ is a discrete substitution of length n, then $A_\theta \in \mathcal{S}\ell_n$.*

PROOF. Let z and z' be elements of T_θ such that $z_i = z'_i$ for $i < N$ and such that $\chi_\theta(z) = \chi_\theta(z')$. Let $m \in (a_N(z), b_N(z))$, $w = z + m\hat{1}$, $w' = z' + m\hat{1}$, and $\omega = \Sigma_{i=0}^{N-1} w_i n^i = \Sigma_{i=0}^{N-1} w'_i n^i$. Pick η and η' bigger then N so that z_η and $z'_{\eta'} \in I$ and set $\xi = \Sigma_{i=0}^{\eta-N} z_{i+N} n^i$ and $\xi' = \Sigma_{i=0}^{\eta'-N} z'_{i+N} n^i$. Then $\chi_\theta(w) = \theta^N(\theta^{\eta-N+1}(0)_\xi)_\omega$ and $\chi_\theta(w') = \theta^N(\theta^{\eta'-N+1}(0)_{\xi'})_\omega$ by Lemma 7 a). Note that ξ and ξ' do not depend upon m, and it suffices to show that $\theta^{\eta-N+1}(0)_\xi = \theta^{\eta'-N+1}(0)_{\xi'}$. But this follows by applying Lemma 7 b) to $\chi_\theta(z) = \chi_\theta(z')$ i.e., $m = 0$ and completes the proof.

Let $x, y \in p_\theta^{-1}(0)$. Because $A_\theta \in \mathcal{S}\ell_n$ it follows that $x \neq y$ if and only if $x(0) \neq y(0)$ or $x(-1) \neq y(-1)$. Suppose $y = \theta(x)$ and $\{z_\alpha\}$ is a sequence in T_θ such that $p_\theta^{-1}(z_\alpha) \rightarrow x$. Then from the proof of Proposition 11 we know that $p_\theta^{-1}(nz_\alpha) \rightarrow y$. It is easy to see that there exist $N > 0$ and $\alpha \in \{0, 1\}$ such that $\theta^{N+1}(\alpha)_0 = x(0)$ and $\theta^{N+2}(\alpha)_0 = y(0)$ from which it follows that $x(0) \neq y(0)$ if and only if $a_0 = 1$ and $b_0 = 0$. By a similar analysis for -1 we get $x(-1) \neq y(-1)$ if and only if $a_{n-1} = 1$ and $b_{n-1} = 0$. Therefore, θ has no fixed points in $p_\theta^{-1}(0)$ if and only if $a_0 = 1$ and $b_0 = 0$ or $a_{n-1} = 1$ and $b_{n-1} = 0$. Moreover, it follows that $\theta^2(x) = x$ for all x in $p_\theta^{-1}(0)$. Thus X_θ is the usual minimal set associated with θ and $p_\theta^{-1}(0)$ contains 1, 2, or 4 points according as $0, n-1 \in I$, $0 \in I$ or $n-1 \in I$ but not both, or $0, n-1 \in J$.

Suppose $0 \in I$. Let $B = A_\theta + \zeta$ where $\zeta = n - 1 + \Sigma_{i=1}^{\infty}(n-2)n^i$ and let ϕ be the substitution given by $\phi(0) = a_1 a_2 \cdots a_{n-1} a_0$ and $\phi(1) = b_1 b_2 \cdots b_{n-1} b_0$. Note that $\phi(\alpha)_{r-1} = \theta(\alpha)_r$ for $r > 0$ and $\theta(\alpha)_0 = \phi(\alpha)_{n-1}$. Let $z = \Sigma_{i=0}^{N} z_i n^i$ with $z_i \in J$ for $i < N$ and $z_N \in I$. Then $\theta^{N+1}(0)_z = 1$ if and only if $z + U_N \subset A_\theta$ if and only if $z + \zeta + U_N \subset B$ if and only if $z' + U_N \subset B$ where $z' = \Sigma_{i=0}^{N} z'_i n^i$, $z'_i = z_i - 1$ for $i < N$, and $z'_N = z_N - 1 \bmod n$. On the other hand, $\theta^{N+1}(0)_z = 1$ if and only if $\phi^{N+1}(0)_{z'} = 1$, and it follows that $B = A_\phi$ and $X_\theta = X_\phi$. Thus we know how to rearrange θ without changing X_θ so that $0 \in J$.

It is obvious that replacing A by $\mathrm{Cl}(Z(n) \setminus A)$ corresponds to replacing X_A by its dual, and it is not hard to show that $\mathrm{Cl}(Z(n) \setminus A_\theta) = Z_\phi$ where

$\phi(i)_r = \theta(i)_r$ if and only if $r \in J$. Hence by at most replacing X_θ by its dual we can assume that $0 \in J$ and $a_{i'} = b_{i'} = 0$ where $i' = \min\{i \notin J\}$. This is precisely the normal form of a discrete substitution as defined by Coven and Keane [2, p. 100].

As a consequence of Theorem 2 and Proposition 12 we now have

THEOREM 4. *If θ is a discrete substitution of length n such that 0 and $n-1$ are not both in J, then (X_θ, σ) is a prime extension of $(Z(n), \hat{1})$.*

EXAMPLE 3: Let $\theta(0) = 0010$ and $\theta(1) = 1011$, and let $\phi(0) = 0010$ and $\phi(1) = 1010$. Then it is easy to check that $A_\phi = A_\theta \cap (\mathrm{Cl}(Z(4) \setminus A_\theta) + \hat{1})$. It follows that A_ϕ can be $\hat{1}$-constructed from A_θ and hence (X_ϕ, σ) is a factor of (X_θ, σ) [8, theor. 2.4]. Since these two minimal sets have very different proximal structures, they can not be isomorphic.

The remainder of this section is devoted to the classification problem. Coven and Keane [2, p. 100] showed that two discrete substitutions of the same lenght produce isomorphic minimal sets if and only if they have the same normal from. We will first show that this result follows from Theorem 1, and then use Theorem 3 to obtain a complete classification of substitution minimal sets on two symbols.

LEMMA 8. *Let θ and ϕ be discrete substitutions in normal form of length n. If $J_\theta = J_\phi$ and if there exists N such that $U_N \cap A_\theta = U_N \cap A_\phi$, then $\theta = \phi$.*

PROOF. Let i_0 be the smallest element in I, and consider $z = i_0 n^{N+1}$. Then $z + U_{N+1} \subset A_\theta$ if and only if $z + U_{N+1} \subset A_\phi$. Hence $\theta^{N+2}(0)_z = 1$ if and only if $\phi^{N+2}(0)_z = 1$. Because $a_{i_0} = 0 = b_{i_0}$ this reduces to $\theta^{N+1}(0)_0 = 1$ if and only if $\phi^{N+1}(0)_0 = 1$, and it follows that $\theta^{N+1}(\alpha)_0 = \phi^{N+1}(\alpha)_0$ by Lemma 7b). Replacing i_0 by any i in I we get $\theta^{N+1}(\theta(0)_i)_0 = \theta^{N+1}(\phi(0)_i)_0$. Since $\theta^{N+1}(\alpha)_0 = \phi^{N+1}(\alpha)_0$ is a bijection of $\{0,1\}$ to $\{0,1\}$, $\theta(0)_i = \phi(0)_i$. Finally let $j \in J$ and repeat the argument with $z = = jn^{N+1} + i_0 n^{N+2}$. This gives $\theta^{N+1}(\theta(0)_j)_0 = \phi^{N+1}(\phi(0)_j)_0$ which implies $\theta(0)_j = \phi(0)_j$. Therefore, $\theta = \phi$.

COROLLARY. *Let θ and ϕ be discrete substitutions of length n. Then $A_\theta = A_\phi$ if and only $\theta = \phi$.*

THEOREM 5. *Let θ and ϕ be discrete substitutions of length n. Then (X_θ, σ) and (X_ϕ, σ) are isomorphic if and only if θ and ϕ have the same normal form.*

PROOF. Suppose (X_θ, σ) and (X_ϕ, σ) are isomorphic. Without loss of generality we can assume that θ and ϕ are in normal form. By Theorem 1, $J_\theta = J_\phi$

and A_θ and A_ϕ are semi-dual, and so there exists N such that either $U_N \cap A_\theta = U_N \cap A_\phi$ or $U_N \cap A_\theta = U_N \cap \mathrm{Cl}(Z(n)\setminus A_\phi)$. It is easy to see that the latter can not hold because they are in normal form. Now we can apply Lemma 8 and conclude that $\theta = \phi$ which completes the proof.

A substitution of length n is called continuous if $\theta(1)$ is the dual of $\theta(0)$, $\theta(0)$ is not constant, and $\theta(0)$ is neither $0101\cdots0$ or $1010\cdots1$ with n odd. A continuous substitution is in normal form if $\theta(0)$ starts with a zero. If θ is a continuous substitution, then it is known that there exists a minimal set X_θ in Ω which contains exactly four sequences such that $\theta^2(x) = x$ and there are no others. Moreover, X_θ is not almost automorphic.

THEOREM 6. *Let θ and ϕ be discrete or continuous substitutions in normal form of length n and m respectively. Then (X_θ, σ) and (X_ϕ, σ) are isomorphic if and only if there exist positive integers p and q such that $n^p = m^q$ and $\theta^p = \phi^q$.*

PROOF. The "if" part is obvious. Suppose that (X_θ, σ) and (X_ϕ, σ) are isomorphic. We distinguish three cases — $|J_\theta| > 1$, $|J_\theta| = 1$, and θ is continuous. For dynamical reasons ϕ must be in the same case as θ. Observe that the first case is an immediate consequence of Theorems 3 and 5. The third case will depend upon the second, and the second requires a careful analysis.

Assume that θ is discrete and $|J_\theta| = 1$ and Ψ is an isomorphism of (X_θ, σ) onto (X_ϕ, σ). It follows that ϕ is discrete, $J_\theta = \{0\} = J_\phi$. If there exist positive integers p and q such that $n^p = m^q$, then by Theorem 5, $\theta^p = \phi^q$ because they are both in normal form. Moreover, we can assume that $\theta(\cdot) = \cdot B$ and $\phi(\cdot) = \cdot C$ when B and C are non-constant $n-1$ and $m-1$ blocks. So it suffices to prove the following lemma:

LEMMA 9. *Let $\theta(\cdot) = \cdot B$ and $\phi(\cdot) = \cdot C$ when B and C are non-constant $n-1$ and $m-1$ blocks of 0's and 1's. If (X_θ, σ) and (X_ϕ, σ) are isomorphic, then there exist positive integers p and q such that $n^p = m^q$.*

PROOF. Let Ψ be an isomorphism of (X_θ, σ) onto (X_ϕ, σ). Replacing Ψ by $\sigma^k \cdot \Psi$ for a suitable k, we can assume that $p_\phi \cdot \Psi = \psi \cdot p_\theta$ when ψ is the canonical isomorphism of $(Z(n), \hat{1}_n)$ onto $(Z(m), \hat{1}_m)$. Let $B = b_1 \cdots b_{n-1}$ and $C = c_1 \cdots c_{m-1}$. For $x \in p_\theta^{-1}(0)$ and $k \neq 0$ we have

$$x(k) = \chi_\theta(k\hat{1}_n) = b_{k'} \tag{1}$$

where k' is the first non-zero coefficient in the n-adic expansion of k. A similar formula holds for ϕ. By Proposition 9 there exists a positive integer N such that $U_N \cap \psi^{-1}(A_\phi)$ equals $U_N \cap A_\theta$ or $U_N \cap \mathrm{Cl}(Z(n)\setminus A_\theta)$. Since

$\mathrm{Cl}(Z(n)\setminus A) = A_{\theta'}$ when $\theta'(\cdot) = \cdot B'$ and B' is dual of B, and since $X_{\theta'}$ is the dual of X_θ, we can assume that $U_N \cap \psi^{-1}(A_\phi) = U_N \cap A_\theta$. It follows that $\Psi(x)(kn^N) = x(kn^N)$ for $x \in p_A^{-1}(0)$ and $k \neq 0$. Notice that the lemma holds for θ and ϕ if and only if it holds for θ^i and ϕ^j. Moreover, θ^i and ϕ^j have the same form as θ and ϕ and thus (1) and its analogue for ϕ hold for θ^i and ϕ^j. We will assume that p and q do not exist and by a succession of replacements of the form θ^i for θ and ϕ^j for ϕ derive a contradiction.

First replace θ by θ^N from which it follows that for $x \in p_\theta^{-1}(0)$ and $k \neq 0$

$$\Psi(x)(kn) = x(kn).$$

Further replacements will not effect this equation.

It is easy to see that there exist positive integers i and j such that $m^i = dn^j$ where d is a positive integer and there exists a prime ρ which divides m and n but not d. Our assumption that p and q do not exist guarantees that $d \neq 1$. Replacing θ by θ^j and ϕ by ϕ^i we have $m = dn$.

There exists an $L > 0$ such that no element of X_ϕ containts a constant block of length greater than or equal to L. Choose β so that $\rho^\beta > L$. Without loss of generality we can assume that ρ^β divides n. Now choose γ so that $d^\gamma > n$, and replace ϕ by ϕ^γ. Hence $m = (d^\gamma n^{\gamma-1})n$ is the new relationship between the lengths of θ and ϕ. For simplicity let $d = d^\gamma$ and then $m = dn^\gamma$ when $\gamma \geqq 1$, $\rho \nmid d$, and $d > n$.

Fix an x in $p_\theta^{-1}(0)$. The goal of the last steps of the proof is to show that C must contain a constant block of length greater than L. Since C appears in $\Psi(x)$, this will be contradiction. Let $d = d_0 + d_1 n + \cdots + d_\alpha n^\alpha$ be the n-adic expansion of d and note that $d_0 \neq 0$ because ρ does not divide d. Clearly $(n, d) = (n, d_0)$ and hence $n/(n, d_0) > L$. Now let $\Gamma = \{1, 2, \cdots, (n/(n, d_0)) - 1\}$. We want to compute c_j for $j \in \Gamma$. First observe that

$$c_j = \Psi(x)(jm) = \Psi(x)(jdn^\gamma) = x(jdn^\gamma) = b_{j'}$$

where j' is the first non-zero coefficient in the n-adic expansion of jd. To compute j' note that $jd = jd_0 + j(d_1 n + \cdots + d_\alpha n^\alpha)$. If $n \mid jd_0$, then $jd_0 = knd_0/(n, d_0)$ which implies that $j \notin \Gamma$. Using $[\,\cdot\,]$ to denote the residue mod n, we now have

$$c_j = b_{[jd_0]}$$

for all j in Γ.

Suppose $1 \leqq i < n - d_0$. So the n-adic expression for $i + d$ is $d_0 + i + d_1 n + \cdots + d_\alpha n^\alpha$. Assume that $m \mid in^\gamma$; i.e., $in^\gamma = km$. It follows that $i = kd > n$ which is impossible. Thus $m \nmid in^\gamma$ and

$b_{i+d_0} = x((i+d)n^\gamma) = \Psi(x)(in^\gamma + m) = \Psi(x)(in^\gamma) = x(in^\gamma) = b_i$. Now suppose that $n - d_0 < i < n$. We can write $i = n - k$ where $1 \leqq k < d_0$ and $[i + d_0] = [n - k + d_0] = d_0 - k$. As above $m \nmid kn$ and

$$b_{d_0-k} = x((d-k)n^\gamma) = \Psi(x)(m - kn^\gamma) = \Psi(x)(-kn^\gamma) = x(-kn^\gamma) = b_i$$

Combining these equations for b_i we have

$$b_i = b_{[i+d_0]}$$

provided that $i \neq n - d_0$. It is easy to check that the first j such that $[jd_0] = n - d_0$ is $n/(n, d_0)$. Therefore, for $j \in \Gamma$

$$c_j = b_{[jd_0]} = b_{d_0}$$

which is the desired contradiction.

For the third case — θ and ϕ discontinuous — we associate with them discrete substitutions θ' and ϕ'. The exact formula for θ' and ϕ' can be found in [2, p. 101] and they are the type with $|J| = 1$. The important facts which are easy to check are that $X_{\theta'}$ is up to isomorphism the only non-equicontinuous factor of X_θ and that θ can be reconstructed from θ'. Thus $X_{\theta'}$ and $X_{\phi'}$ are isomorphic and there exist integers p and q such that $n^p = m^q$. It follows that θ^p and ϕ^q produce isomorphic minimal sets and then $(\theta^p)'$ and $(\phi^q)'$ have the same normal form. From here one checks that θ^p either equals ϕ^q or its dual, and since θ^p and ϕ^q are in normal form, θ^p must equal ϕ^q.

Theorem 6 together with Theorem 5 provides a finite process for determining whether or not two substitutions produce isomorphic minimal sets. Let θ and ϕ be two substitutions of length m and n. Without loss of generality we can assume they are in normal form. First factor their lengths into primes. Unless exactly the same primes appear in both n and m the minimal sets are not isomorphic. For each prime appearing in n (and m) find the ratio of the number of times it appears in n to the number of times it appears in m. Unless this ratio is the same for all primes appearing in n the minimal sets are not isomorphic. Assuming this ratio is constant, let p be the denominator and let q be the numerator. Then $n^p = m^q$ and both θ^p and ϕ^q are in normal form. Unless $\theta^p = \phi^q$ the minimal sets are not isomorphic by Theorem 5.

Example 4. Let $n = 4$, let $\theta(0) = 0101$ and $\theta(1) = 1001$, let $\phi(0) = 0001$ and $\phi(1) = 1101$, and let $A = (A_\theta \cap U_0) \cup (A_\phi \cap \hat{1} + U_0)$. It is easy to check that $A \in \mathcal{S}\ell_4$ with $J_A = J_\theta = J_\phi = \{0, 1\}$. Using the kind of argument in the proof of Lemma 8, it can be shown that if (X_A, σ) is isomorphic to a substitution η in normal form, then $\theta^p = \phi^p$ for some p. Therefore, there are substitution-like minimal sets which are not isomorphic to any substitution minimal set.

References

1. E. Coven, *Endomorphisms of substitution minimal sets*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **20** (1971), 129–133.

2. E. Coven and M. Keane, *The structure of substitution minimal sets*, Trans. Amer. Math. Soc. **162** (1971), 89–102.

3. W. Gottschalk, *An irreversible minimal set*, in *Ergodic Theory, Proceedings of an International Symposium* (New Orleans, 1961), Academic Press, New York, 1963, pp. 121–134.

4. W. Gottschalk, *Substitution minimal sets*, Trans. Amer. Math. Soc. **109** (1963), 467–491.

5. K. Jacobs and M. Keane, 0-1-*Sequences of Toeplitz type*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **13** (1969), 123–131.

6. H. Keynes, *The proximal relation in a class of substitution minimal sets*, Math. Systems Theory **1** (1967), 165–174.

7. B. Klein, *Homomorphisms of symbolic dynamical systems*, Math. Systems Theory **6** (1972), 107–122.

8. N. Markley, *Characteristic sequences*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete (to appear).

9. J. Martin, *Substitution minimal flows*, Amer. J. Math. **93** (1971), 503–526.

10. J. Martin, *Minimal flows arising from substitutions of non-constant length*, Math. Systems Theory **7** (1973), 73–82.

11. M. Morse, *Recurrent geodesics on a surface of negative curvature*, Trans.Amer. Math.Soc. **22** (1921), 84–100.

12. M. Morse, *Symbolic Dynamics, Lectures by Marston Morse*, 1937–1938. *Notes by Rufus Oldenburger*, The Institute for Advanced Study, Princeton, New Jersey, 1966.

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